Bisector Graphs for Min-/Max-Volume Roofs over Simple Polygons

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Abstract

Piecewise-linear terrains ("roofs") over simple polygons were studied by Aichholzer et al. (1995) in their work on straight skeletons of polygons. We show how to construct a roof over a simple polygon that has minimum (or maximum) volume among all roofs that drain water. Such a maximum-volume (minimum-volume) roof can have quadratic (maybe cubic, resp.) number of facets. Our algorithm for computing such a roof extends the standard wavefront propagation known from the theory of straight skeletons by two additional events. Both the minimum-volume and the maximum-volume roof of a simple polygon with n vertices can be computed in $\mathcal{O}(n^3 \log n)$ time.

1 Introduction

1.1 Motivation and Prior Work

In 1995 Aichholzer et al. [3] introduced straight skeletons of simple polygons. Their work also highlights the intimate connection between straight skeletons — as a special form of a bisector graph — of polygons in the two-dimensional plane and a 3D structure called "roof". The bisector graph and the roof model are used to demonstrate straight skeleton properties. They mention that its roof volume is neither maximized nor minimized. Their algorithm uses a sweep-plane approach to compute the straight skeleton of a simple polygon in $\mathcal{O}(n^2 \log n)$ time. Aichholzer and Aurenhammer [2] apply a wavefront propagation to compute straight skeletons of general planar straight-line graphs.

While every straight skeleton of a simple polygon has its corresponding roof [3], it seems natural to study also other types of roofs. Indeed, so-called "realistic roofs" were introduced in recent work by several authors [6, 1]. Their approach enumerates all possible realistic roofs over a rectilinear polygon in $\mathcal{O}(n^5)$ time. A side result of their work is the computation of a realistic roof that has minimum height or minimum volume (under the roof).

We pick up this lead and generalize realistic roofs to "natural roofs": Roughly, we still require a natural roof to drain water but wave the restriction that every facet of the roof has to be connected to its defining boundary edge. (See the gray triangular area in Fig. 1b.) We show how to employ a wavefront propagation to compute a minimum-volume (maximum-volume) roof of a simple polygon with n vertices in $\mathcal{O}(n^3 \log n)$ time.

1.2 Basics

Throughout this paper we let P denote a simple polygon in the xy-plane, Π_0, of $\mathbb{R}^3$. The *interior side* of an edge e of P is the half-plane (within Π_0) induced by its supporting line $\ell(e)$ which locally (close to e) overlaps with the interior of P. We associate a half-plane $\Pi(e)$ with e in the following way: (i) The intersection of $\Pi(e)$ with Π_0 is given by $\ell(e)$, (ii) $\Pi(e)$ lies within the half-space $z \geq 0$ of $\mathbb{R}^3$, (iii) the normal projection of $\Pi(e)$ onto Π_0 coincides with the interior side of e; i.e., $\Pi(e)$ is inclined towards the interior of P, and (iv) $\Pi(e)$ forms a 45° angle with Π_0.

Consider two different edges e_1, e_2 of P. The (angular) *bisector* of e_1, e_2 is the set of all points within the intersection of the interior sides of e_1 and e_2 that are equidistant from $\ell(e_1)$ and $\ell(e_2)$.

For the sake of (mostly descriptional) simplicity we assume that P is in general position: (i) No two edges of P are parallel to each other, and (ii) not more than three bisectors of edges of P meet in one point. Under this assumption, the bisector of two edges e_1, e_2 of P is a ray that starts at the point of intersection $\ell(e_1) \cap \ell(e_2)$ and leads into the common interior of e_1 and e_2.

Definition 1 (Bisector Graph [3]) *A connected planar straight-line graph is a* bisector graph, $\mathcal{B}(P)$, *of P if (i), all its edges are portions of bisectors of edges of P, (ii) it has no degree-two node, and (iii) there is a bijection between its degree-one nodes and the vertices of P.*

The straight skeleton of P is known to be one specific bisector graph of P [3]; cf. Fig. 1a and 1b. The edges of a bisector graph are called $\mathcal{B}$-arcs, and common end-points of $\mathcal{B}$-arcs are called $\mathcal{B}$-nodes. By letting a $\mathcal{B}$-arc of $\mathcal{B}(P)$ inherit the orientation of its supporting bisector ray we impose an orientation onto every $\mathcal{B}$-arc, thus turning a bisector graph into a *directed bisector graph*. Naturally, $\mathcal{B}$-nodes in a directed bisector graph have an in-degree and out-degree.

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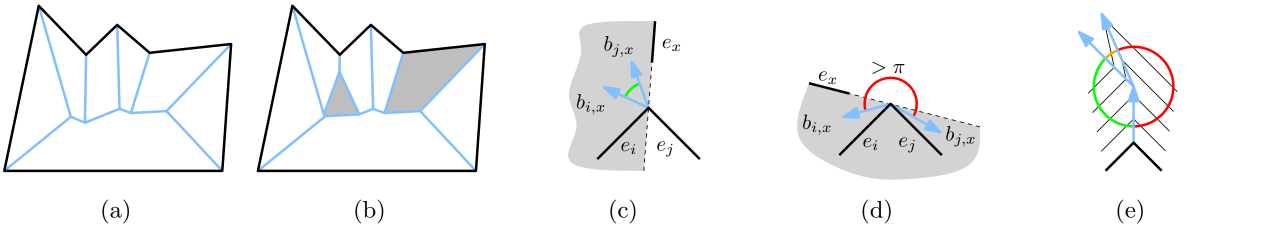

Figure 1: In (a), we see the straight skeleton of a simple polygon P, in (b) we see another bisector graph of P. In (c), we see a create event while in (d), due to an internal angle greater than π (red), no create event is given. In (e), we see how a create event modifies the wavefront polygon.

Definition 2 (Roof Model [3]) *A* roof *for* P, $\mathcal{R}(P)$, *is a terrain over* P, *i.e., the graph of a piecewise-linear continuous function over* P, *such that (i) every facet of* $\mathcal{R}(P)$ *is a maximal connected subset of a half-plane* $\Pi(e)$ *of some edge* e *of* P, *and (ii) the intersection of* $\mathcal{R}(P)$ *with* Π_0 *is equal to the boundary of* P.

Theorem 1 ([3]) *Every roof for* P *corresponds to a unique bisector graph of* P, *and vice versa.*

We say that an *edge* e of P *defines a facet* f of $\mathcal{R}(P)$ if f is contained in $\Pi(e)$. Note that some edge may define multiple facets. As usual, a vertex v of P is called reflex if the internal angle at v is greater than π; convex otherwise. We call an edge e between two neighboring facets of a roof *valley* or *ridge* depending on whether e originates from a reflex or convex vertex.

Consider a facet f of a roof of P, and let f' be its normal projection onto Π_0. The *truncated prism defined by* f is the solid bounded by f, f' and by trapezoids between all pairs of corresponding edges of f and f'.

If all facets of the roof of a house have the so-called gradient property then water is guaranteed to drain and local minima are omitted [3]. Since the gradient property requires every point on a facet of an edge e to have a steepest path to e this is a sufficient but not a necessary condition condition for water to drain. We consider a less stringent requirement for a roof to drain water and still omit local minima.

Definition 3 (Natural Gradient Property) *Let* $\mathcal{R}(P)$ *be a roof for* P. *We say that a facet* f *of* $\mathcal{R}(P)$ *has the* natural gradient property (NGP) *if, for every point* $p \in f$, *there exists a path* $g(p)$ *that (i) starts at* p, *(ii) follows the steepest gradient, and (iii) reaches the boundary of* P.

Definition 4 (Natural Roof) *A roof* $\mathcal{R}(P)$ *for a polygon* P *is called* a natural roof for P *if all its facets have the natural gradient property.*

Definition 5 (Min-/Max-Vol. Bisector Graph) *The* maximum-volume bisector graph $\mathcal{B}_{\max}(P)$ *of a polygon* P *is a bisector graph* $\mathcal{B}(P)$ *whose associated roof* $\mathcal{R}(P)$ *is a natural roof that maximizes the volume over all possible natural roofs for* P. *Similarly for the* minimum-volume bisector graph $\mathcal{B}_{\min}(P)$.

Definition 6 (Capped Roof) *A* capped (natural) roof *with height* $t \geq 0$ *for a polygon* P *is the set of all points of a (natural) roof* $\mathcal{R}(P)$ *of* P *whose* z*-coordinate does not exceed* t.

2 Computing Min-/Max-Volume Roofs

Wavefront propagation [3, 2] is a well-known strategy for computing straight skeletons. Roughly, the wavefront propagation of P is a shrinking process in which every input edge of P is offset inwards in a self-parallel manner. Initially, the segments of the wavefront correspond to the edges of the polygon. During the wavefront propagation every wavefront segment moves at unit speed towards the interior of P. It is common to regard the wavefront as a function of time t and to write $\mathcal{W}_P(t)$ to denote the shrinking (wavefront) polygons at time t. At time t every wavefront segment is at normal distance t from its input edge. As time progresses, the normal distance of each wavefront segment to its defining input edge grows. The points of intersection between consecutive wavefront segments lie on the bisectors of their defining input edges.

The wavefront vertices move along these bisectors and trace out the desired bisector graph. In order to maintain the planarity of $\mathcal{W}_P(t)$, and to obtain a straight skeleton, one has to handle two events: An *edge event* occurs when a wavefront segment shrinks to zero length. A *split event* occurs when a wavefront vertex crashes into a wavefront edge which moves in the opposite direction. A split event results in the split of the wavefront polygon into two sub-polygons.

Observation 1 (Vertex Speed [4]) *The speed of a wavefront vertex* v *is given by* $s(v) = \frac{1}{\sin(\alpha/2)}$, *where* α *is the exterior angle of* v.

We will also employ a wavefront propagation to compute a min-/max-volume bisector graph. Our process involves four event types: *edge event*, *split event*, as in the straight skeleton computation [3], and

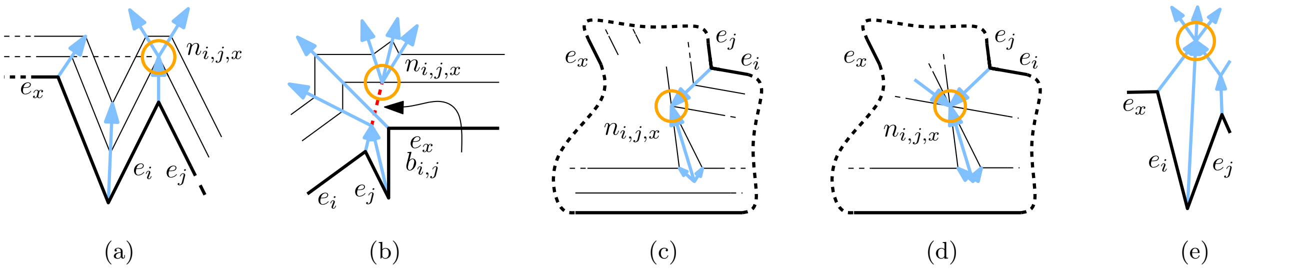

Figure 2: In (a–b), we see create events and in (c–d) we see divide events; and (e) shows a bisector graph that is not produced by our propagation scheme. Note that the (red) dashed line in (b) is part of the bisector $b_{i,j}$ on which a stealth vertex moves.

divide event and *create event* as two new event types. In addition, we employ so-called *stealth vertices*.

Definition 7 (Stealth Vertex) *Let $p_{i,j} := \ell(e_i) \cap \ell(e_j)$ be the point of intersection of the supporting lines of two edges e_i, e_j of P. As the wavefront propagates, $p_{i,j}$ moves inwards along the bisector $b_{i,j}$ of e_i, e_j, at a speed given by Observation 1. At any time when $p_{i,j}$ is not part of the wavefront polygon we call it a* stealth vertex *of P. (See Fig. 2b.)*

Definition 8 (Create Event) *If (1) the supporting line of a wavefront edge e becomes incident with a reflex wavefront vertex v, where e is not incident at v, or (2) a stealth vertex v becomes incident with a wavefront edge e, and for either (1) or (2) the interior angle between the two bisectors between e and the two edges incident at v is smaller than π, then we call it a* create event*. (See Fig. 1c and 1d.)*

In the former case we insert an edge with zero length between the two edges defining the reflex vertex. The new edge belongs to the same input edge as the one defining the supporting line; cf. Fig. 2a. In the latter case we insert two edges with zero length at the point of intersection, thus splitting the intersected wavefront edge. The two edges associated with the stealth vertex define the two new edges and the stealth vertex becomes a wavefront vertex; cf. Fig. 2b.

We recall that the general position of P prohibits parallel input segments. However, according to its definition a create event adds parallel wavefront segments to a wavefront polygon. This is a necessary condition for the *divide event*.

Definition 9 (Divide Event) *When two or three reflex wavefront vertices become incident and all incident wavefront edges originate from three common input edges we call it a* divide event*.*

In the former case the two parallel edges (associated with one input edge) join into one edge and the two remaining edges become adjacent; cf. Fig. 2c. In the latter case three parallel edge pairs become incident and all edges change adjacencies to their neighbor; cf. Fig. 2d.

We point out that the divide event is not a "vertex-event" [5] in disguise where reflex wavefront vertices become incident as well. Note that all events but the create event are compulsory: Ignoring only one of them during the wavefront propagation would result in a self-intersecting wavefront. Only create events are optional: Accepting or ignoring such an event gives us the freedom to construct different roofs and, thus, to influence the volume of the resulting roof.

Every event takes place at the intersection of three bisectors and forms a $\mathcal{B}$-node in the bisector graph. Three $\mathcal{B}$-arcs start or end at every $\mathcal{B}$-node, except for three cases: (i) at the vertices of P; (ii) degree-six nodes that occur in a divide event where three reflex wavefront vertices become incident; cf. Fig. 2d; and (iii) another degree-six node which is not listed as an event; cf. Fig. 2e. However, in (i) no event takes place and in (ii) the in-degrees and the out-degrees both are three and the directions of the incident $\mathcal{B}$-arcs alternate when one moves around the node. Lastly, (iii) is not relevant in our propagation schema as it can be omitted due to the goal of min-/maximizing.

Besides these exceptions, no two $\mathcal{B}$-arcs can lie on a common bisector and intersect at a common $\mathcal{B}$-node. In the full paper we list all combinatorically possible bisector embeddings for a $\mathcal{B}$-node and show that every combination is considered.

By definition, all events occur at intersections of bisectors of P. Furthermore, before and after each event all wavefront vertices advance on bisectors of P. The proofs of the subsequent claims are contained in the full paper.

Lemma 2 *Any wavefront propagation results in a bisector graph.*

Lemma 3 *Any wavefront propagation results in a roof.*

Lemma 4 *A capped roof of height t, constructed using a wavefront propagation, fulfills the natural gradient property for all its facets.*

Lemma 5 *A bisector graph can be seen as a directed acyclic graph (DAG).*

Lemma 6 *The speed of a wavefront vertex v defines the slope of its associated ridge or valley, with respect to v and Π_0.*

Corollary 7 *A slower wavefront vertex leads to a locally larger slope of its associated ridge or valley, and vice versa.*

Lemma 8 *The volume of a natural roof created by a wavefront propagation can only be influenced by a create event.*

Lemma 9 *A create event takes place at a reflex wavefront vertex p. A small disc c centered at p is partitioned into three wedges by the three $\mathcal{B}$-arcs incident at p. If one wedge has an angle greater than π it involves a wavefront vertex, starting at p, that moves faster than the wavefront vertex which ends at p; cf. Fig. 1e.*

Definition 10 (De-/Accelerating Create Event) *Accepting a create event during the wavefront propagation results in new wavefront vertices. If one of the new wavefront vertices moves faster than the intersected wavefront edge or vertex then we call it an "accelerating" create event. If all new wavefront vertices move slower than the intersected wavefront vertex then we call it a "decelerating" create event.*

Lemma 10 *A create event with out-degree three is always an accelerating create event.*

Note that this implies that decelerating create events can only occur on reflex wavefront vertices, i.e., for the first case of Definition 8(1). Furthermore, Lemma 9 implies that Definition 10 is complete; there is no third class of create events.

Lemma 11 *A decelerating (accelerating) create event increases (decreases, resp.) the roof volume.*

Summarizing, a faster moving vertex increases the area that is swept by the wavefront, thus resulting in a locally reduced roof volume. Other propagation events that occur earlier can, at most, lead to a reduction of the roof volume. Conversely for a slower moving vertex. Hence, one run of the wavefront propagation (without backtracking) suffices to obtain a minimum-volume or maximum-volume roof. We summarize our result in the following theorem.

Theorem 12 *Accepting all decelerating (accelerating) create events during the wavefront propagation leads to $\mathcal{B}_{\max}$ ($\mathcal{B}_{\min}$, respectively).*

3 Analysis

We use a wavefront propagation for both $\mathcal{B}_{\min}(P)$ and $\mathcal{B}_{\max}(P)$. The complexity is dominated by the computation of all create events. For $\mathcal{B}_{\max}(P)$, one reflex input vertex can result in $\mathcal{O}(n)$ create events. Thus, we can get a quadratic number of facets. To handle one create event during the propagation takes $\mathcal{O}(n \log n)$ time. The overall complexity is therefore $\mathcal{O}(n^3 \log n)$ time and $\mathcal{O}(n^2)$ space. For $\mathcal{B}_{\min}(P)$, it is unclear if more than $\mathcal{O}(n^2)$ create events can occur. Hence, $\mathcal{B}_{\min}(P)$ may admit a cubic number of facets which leads to $\mathcal{O}(n^3)$ space, but the same time complexity.

4 Extensions

Our natural roofs can be regarded as a generalization of realistic roofs [6, 1], but our current definitions prevent a clean mathematical statement regarding any subset relation among these two types of roofs: The work of [6, 1] is restricted to rectilinear polygons, while we exclude parallel input edges explicitly. However, since parallel segments might occur during the wavefront propagation, we suspect that the restriction on the input edges can be waived. This would permit to apply our approach to the setting of [6, 1], thus reducing the time complexity for finding a minimum-volume realistic roof from $\mathcal{O}(n^5)$ to $\mathcal{O}(n^3 \log n)$. Obtaining a minimum-height roof is ongoing work, too.

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